Capacity of the Hopfield model

1997 J. Phys. A: Math. Gen. 30 3383

(http://iopscience.iop.org/0305-4470/30/10/016)

J. Phys. A: Math. Gen. **30** (1997) 3383–3391. Printed in the UK PII: S0305-4470(97)78152-6

Capacity of the Hopfield model

Jianfeng Feng† and Brunello Tirozzi‡

† Statistics Group, The Babraham Institute, Cambridge CB2 4AT, UK and Mathematisches Institut, Universität München, D-80333 München, Germany

‡ Mathematical Department, University of Rome 'La Sapienza', Rome, Italy

Received 26 September 1996, in final form 23 December 1996

Abstract. For a given $0 < \delta < \frac{1}{2}$ if $[N\delta]$ neurons deviating from the memorized patterns are allowed, we constructively show that if and only if $\alpha(\delta) := p(N)/N = (1-2\delta)^2/(1-\delta)^2$ all stored patterns are *fixed points* of the Hopfield model. If $[N\delta_N]$ neurons are allowed with $\delta_N \to 0$ then $\alpha_N = (1-2\delta_N)^2/(\Phi^{-1}(1-\delta_N))^2 \to 0$ where Φ is the distribution function of the normal distribution. The result obtained by Amit and co-workers only formally coincides with the latter case which indicates that the replica trick approach to the capacity of the Hopfield model is only valid in the case $\alpha_N \to 0 (N \to \infty)$.

1. Introduction

Although a number of mathematical neural network models were known before, one proposed by Hopfield in 1982 has provoked a great interest in the scientific community. This is a mathematical model aiming to describe a network functioning as an associative memory, i.e. it is able to store information and retrieve it when given degraded data [1, 3, 8].

The Hopfield model includes a set of N formal binary neurons labelled by elements of $\{1, \ldots, N\}$ and interconnected one by one. The state of the ith neuron is described by a spin variable σ_i(neuron activity) with values in $\{-1, 1\}$, and the whole network by a configuration of spins $\sigma \in X := \{-1, 1\}^N$, $\sigma = (\sigma_i)_{i=1,\ldots,N}$. This network is designed to memorize $p(N)$ patterns $\xi^{(\mu)} \in X$, $\mu = 1, \ldots, p(N)$ in the following sense: one associates the pattern $\xi^{(\mu)}$ to any configuration sufficiently 'close' to it. 'Close' usually means a small Hamming distance.

The retrieval process is described by a dynamics on the configuration space X given by

$$\sigma_i(t+1) = \mathrm{sign}\left(\sum_{j=1}^{N} T_{ij}\sigma_j(t)\right) \qquad t = 1, \ldots \tag{1}$$

and so a configuration $\sigma = (\sigma_i, i = 1, \ldots, N)$ is a fixed point of the dynamics of (1) if and only if

$$\sigma_i\left(\sum_{j=1}^{N} T_{ij}\sigma_j\right) \geqslant 0 \qquad i = 1, \ldots, N. \tag{2}$$

This dynamics can be realized either synchronously or asynchronously [6, 7, 10]. The idea on the choice of the connection T_{ij} between the ith neuron and the jth neuron is based upon the so-called Hebb learning rule, i.e.

$$T_{ij} = \frac{\sum_{\mu=1}^{p(N)} \xi_i^{(\mu)}\xi_j^{(\mu)}}{N} \qquad i, j = 1, \ldots, N. \tag{3}$$

The properties of the Hopfield model have made it an interesting candidate for theoretical studies, for models of some brain functions and also for technical applications in certain areas of computer development or artificial intelligence. In all cases, one of the first questions that comes to mind is the storage capacity of such a model, namely the quantity of information that can be stored and effectively retrieved from the network. It is of primary interest to know, for example, how the number of patterns varies with the number of neurons. We cited below some important properties of the memory capacity that hold with probability approaching 1 as $N \to \infty$.

It has been rigorously shown by MacEliece *et al* [18] that if $p(N) < N/(4\log N)$ then the *exact* original patterns are attractors of the dynamics of (1). This result has been extended by Komlos and Paturi [13] who have shown that each pattern has a non-trivial radius of attraction. Numerical computations and heuristic arguments given in [18] suggest that $p(N)$ can only grow proportionally to $N/(\log N)$ in order for every pattern to be exactly recoverable.

If a small fraction of errors is tolerated in retrieved patterns, then $p(N)$ can increase more rapidly. Nevertheless, it has been previously suggested by numerical simulations, as well as by not fully rigorous analytical computations, that there is a collapse in the model functioning in the regime $p(N)/N \to \alpha > 0$ as $N \to \infty$. Hopfield [11] has studied his model numerically for large values of N with $p(N)/N$ near some α and with the patterns chosen at random. He has observed that the memory only functioned well for values of α less than 0.1. The nature of this collapse was clarified in a paper by Amit *et al* [2], who introduced a 'thermodynamic' version of the model and discovered that by using the *non-rigorous* replica method for $\beta \geqslant 0$ the critical capacity is about 0.15. There are also a number of mathematical results concerning this models, see for example [20] and references therein. In particular, in [19] the authors tried to replace the replica symmetric trick by the assumptions on the self-average properties of the Edwards–Anderson parameter and the very results from [2] are re-obtained.

The first rigorous proof of the existence of an energy barrier for small α was given by Newman in [16]. This result was later extended to the case where $\alpha \geqslant 0.071$ [14]. However, due to the complexity of the energy landscape they are not all able to reach the conclusion that dynamics equation (1) with an initial configuration in a ball, centre at stored pattern and with small enough radius, will not exit from this ball.

In this paper we present a transparent approach based upon the extreme value theory of statistics and rigorously obtain the capacity of the Hopfield model to store $p(N)$ patterns as fixed points.

2. Extreme value theory

As usual we assume that $P(\xi_i^{(\mu)} = 1) = \frac{1}{2}$, $\xi_i^{(\mu)}, \mu = 1, \ldots, p, i = 1, \ldots, N$ are independant identically distributed (iid) random variables taking values in $\{-1, 1\}$ and for a set $A \subset \{1, \ldots, N\}$ we have, by the central limit theorem,

$$g(N, p(N)) := \xi_i^{(1)} \left(\frac{1}{N} \sum_{\mu=2}^{p(N)} \sum_{j=1, j\neq i} \xi_i^{(\mu)} \xi_j^{(\mu)} \xi_j^{(1)} - 2\frac{1}{N} \sum_{\mu=2}^{p(N)} \sum_{j\in A, j\neq i} \xi_i^{(\mu)} \xi_j^{(\mu)} \xi_j^{(1)} \right) \to -\sqrt{\frac{p(N)}{N}} \zeta_i \qquad (4)$$

weakly where $\zeta_i, i = 1, \ldots, N$ are iid normally distributed random variables with mean 0 and covariance 1. Note that the convergence in equation (4) is independent of the choice of set A. Let ζ_{Nk} be the $(N-k)$th largest maximum and hence $\zeta_{NN} = \max_{1\leqslant i\leqslant N}\zeta_i$, the largest maximum. In the sequel for the simplicity of notation we take the convention that neurons are numbered according to the increasing order of ζ, namely $Ni = i$. Let $[a]$ be the integer part of $a \in R^1$. For $0 \leqslant x \leqslant 1$ the behaviour of $\zeta_{[xN]}$ is exactly known in the extreme value theory of statistics (see figure 1):

(i) $x = 1$—we have $\zeta_N = \max_{1\leqslant i\leqslant N}\zeta_i$ and (see [15, p 15])

$$\lim_{N\to\infty} P(e_N(\zeta_N - g_N) \leqslant x) = \exp(-\mathrm{e}^{-x}) \tag{5}$$

with

$$e_N = (2\log N)^{\frac{1}{2}}$$

and

$$g_N = (2\log N)^{\frac{1}{2}} - \tfrac{1}{2}(2\log N)^{-\frac{1}{2}}(\log\log N + \log 4\pi) \sim e_N$$

which enables us to deduce that

$$\zeta_N \to \epsilon_{g_N} \tag{6}$$

weakly where ϵ_a represents the Dirac measure at point a.

(ii) $0 < x < 1$—we know from corollary 4 to theorem 5.7 in [4, 5] that

$$\lim_{N\to\infty} P(a_N(\zeta_{[xN]} - x) \leqslant z) = \int_{-\infty}^{z} \mathrm{e}^{-\frac{y^2}{2}}\,\mathrm{d}y\,\frac{1}{\sqrt{2\pi}} = \Phi(z) \tag{7}$$

with

$$a_N = \sqrt{\frac{N^3}{[xN](N-[xN])}} \tag{8}$$

and therefore

$$\zeta_{[xN]} \to \epsilon_x \tag{9}$$

weakly since $a_N \to \infty$ when $N \to \infty$.

(iii) $x_N < 1$ but $x_N \to 1$ and $N\sqrt{1-x_N} \to \infty$—theorem 3.3 in [21] tells that

$$\lim_{N\to\infty} P(c_N(\zeta_{[Nx_N]} - d_N) \leqslant z) = \Phi(z) \tag{10}$$

with

$$\Phi(d_N) = x_N \qquad c_N = N\Phi'(d_N)/\sqrt{(1-x_N)} \tag{11}$$

and therefore

$$\zeta_{[Nx_N]} \to \epsilon_{d_N} \tag{12}$$

weakly since

$$\begin{aligned}
\lim_{N\to\infty} c_N &= \lim_{N\to\infty} (N d_N(1-\Phi(d_N)))/\sqrt{1-x_N} \\
&= \lim_{N\to\infty} N\Phi^{-1}(x_N)\sqrt{1-x_N} \\
&= \lim_{N\to\infty} (N\sqrt{1-x_N})/(\Phi'(\Phi(d_N))) = \infty
\end{aligned}$$

where we have applied the following relation

$$\text{for } d_N \to \infty \text{ we have } 1-\Phi(d_N) = \Phi'(d_N)/d_N. \tag{13}$$

Our following developments totally rely on the behaviours of $\zeta_{[xN]}$: when $x = 1$, ζ_N goes to infinity in the order of $\sqrt{2\log N}$; when $x_N \to 1$, $\zeta_{[x_N N]}$ tends to infinity to the

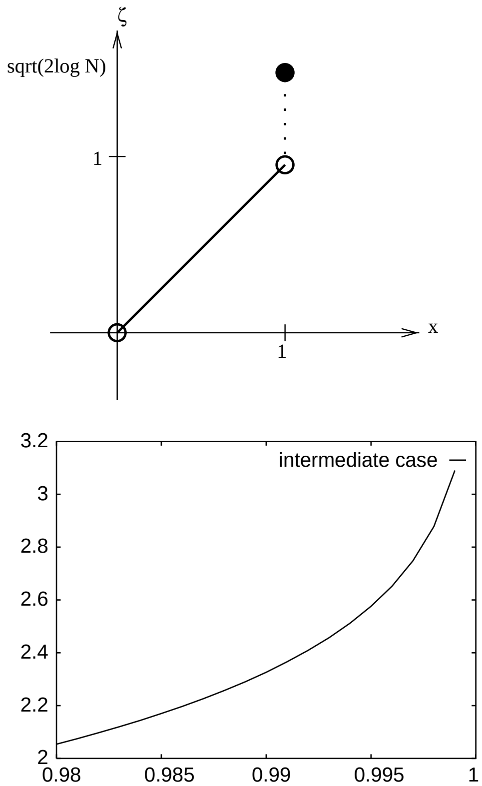

Figure 1. $\zeta_{[xN]}$ as a function of x. Note that when $x = 1$, $\zeta_N \sim \sqrt{2\log N}$ goes to infinity ($N \to \infty$) and when $0 < x < 1$, $\zeta_{[xN]} \sim x < 1$. The intermediate case $x_N \to 1$, $x_N < 1$ described by equation (12) is reploted.

order of $\Phi^{-1}(x_N)$; finally when $0 < x < 1$, $\zeta_{[xN]}$ stays at x. These differences embody the different behaviours of the capacity of the Hopfield model discussed in the next section. In figure 1 we plot $\zeta_{[xN]}$ as a function of x, $0 < x \leqslant 1$. At $x = 1$, $\zeta_{[xN]}$ is discontinuous; it jumps from a finite value to the value $(2\log N)^{\frac{1}{2}}$. Due to the difference among the cases $0 < x < 1$, $x = 1$ and $x_N \to 1$ we take into account these three cases separately. Let us start from the simplest case $x = 1$ corresponding to $\delta = 0$.

3. Capacity

3.1. $\delta = 0$*—perfect retrieval*

From equation (4) we obtain

$$\xi_i^{(1)}\left(\frac{1}{N}\sum_{\mu=1}^{p(N)}\sum_{j=1,j\neq i}^{N}\xi_i^{(\mu)}\xi_j^{(\mu)}\xi_j^{(1)}\right) = \xi_i^{(1)}\left(\frac{1}{N}\sum_{j=1,j\neq i}^{N}\xi_i^{(1)}\xi_j^{(1)}\xi_j^{(1)} + \frac{1}{N}\sum_{\mu=2}^{p(N)}\sum_{j=1,j\neq i}^{N}\xi_i^{(\mu)}\xi_j^{(\mu)}\xi_j^{(1)}\right)$$
$$= 1 - g(p(N), N)$$
$$\to 1 - \sqrt{\frac{p(N)}{N}}\zeta_i \qquad (14)$$

for $i = 1, \ldots, N$. To ensure that $\xi^{(1)}$ is a fixed point of the dynamics of (1) if and only if

$$1 - \sqrt{\frac{p(N)}{N}}\zeta_N \geqslant 0 \tag{15}$$

since $\zeta_i \leqslant \zeta_N$ and

$$1 - \sqrt{\frac{p(N)}{N}}\zeta_N \leqslant 1 - \sqrt{\frac{p(N)}{N}}\zeta_i \tag{16}$$

for $i \in \{1, \ldots, N\}$. From equation (6) we deduce that

$$1 - \sqrt{\frac{p(N)}{N}}\zeta_N \to 1 - \sqrt{\frac{p(N)}{N}}\sqrt{2\log N}. \tag{17}$$

Combining equation (15) and (17) we conclude that $\xi^{(1)}$ is a fixed point of the dynamics of (4) if and only if

$$p(N) \leqslant p_c(N) = N/(2\log N). \tag{18}$$

• Result (18) corresponds to the trivial case $\alpha = 0$ and is well known (see for example [3, 18]). The very result is directly obtainable from the statistical signal to noise analyses (see [3, pp 278–82]).

• Our analyses above indicate that in the simplest case the capacity is utterly determined by the largest maximum of ζ.

3.2. $\delta > 0$—*retrieval with a fixed error tolerance*

Different from the situation discussed above for $\delta = 0$, here we have the freedom to reverse the sign of the activity of $[N\delta]$ neurons and the capacity, as one may expect, is not solely dependent on ζ_N. We address the following two rudimentary issues:

• How many patterns, $p(N)$, can we store in a network so that a configuration $\sigma(\xi^{(1)})$ with the property

$$m(\sigma(\xi^{(1)}), \xi^{(1)}) = \left(\sum_i \sigma_i(\xi^{(1)})\xi_i\right)\bigg/ N = 1 - 2\delta$$

is a fixed point of the dynamics of (1)?

• Furthermore, how do we choose a set B, with $[N\delta]$ neurons in B, and satisfy the property that

$$\sigma_i(\xi^{(1)}) = -\xi_i^{(1)} \qquad i \in B?$$

As we have perceived from the discussion above for $\delta = 0$, those neurons taking extreme values of ζ play an important role for the model. More exactly,we are going to prove that

$$\sigma(\xi^{(1)}) := (\xi_1^{(1)}, \xi_2^{(1)}, \ldots, \xi_{[(1-\delta)N]}^{(1)} - \xi_{[(1-\delta)N]+1}^{(1)}, \ldots, -\xi_N^{(1)}) \tag{19}$$

is a fixed point of the dynamics of (1). Therefore $B = \{[(1-\delta)N] + 1, \ldots, N\}$. In other words for a given $\delta > 0$, $\sigma(\xi^{(1)})$ is a fixed point of the dynamics of (1) if and only if $\sigma(\xi^{(1)})$ is the configuration which reverses the sign of $\xi_i^{(1)}$ with ζ_i taking the $(N-i)$th largest maximum of ζ where $i > [(1-\delta)N]$.

Next let us check (see equation (2)) that

$$\sigma_i(\xi^{(1)})\left(\sum_{j=1, j\neq i}^{N} T_{ij}\xi_j^{(1)} - 2\sum_{j\in B, j\neq i} T_{ij}\xi_j^{(1)}\right) \geqslant 0 \tag{20}$$

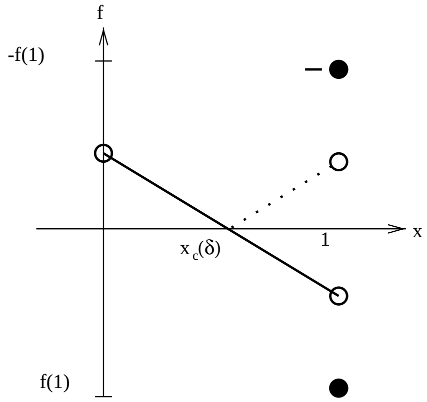

Figure 2. f defined by equation (22) (full line). $x_c(\delta) = (1-\delta)$ and filled circle$= f(1) = 1-2\delta-\sqrt{p(N)/N}\sqrt{2\log N}$ going to negative infinity ($N \to \infty$) if $\alpha > 0$. Broken line $= -f(x)$, $x_c(\delta) < x \leqslant 1$.

for all $i = 1, \ldots, N$. To this end we define a function† for $0 < x \leqslant 1$ by

$$f(x) = \begin{cases} 1 - 2\delta - \sqrt{\alpha}x & \text{if } 0 < x < 1 \\ 1 - 2\delta - \sqrt{\alpha}\sqrt{2\log N} & \text{if } x = 1. \end{cases} \tag{21}$$

Combining (4), (6) and (9) we get that

$$\xi_i^{(1)}\left(\sum_{j=1, j\neq i}^{N} T_{ij}\xi_j^{(1)} - 2\sum_{j\in\{[(1-\delta)N]+1,\ldots,N\}, j\neq i} T_{ij}\xi_j^{(1)}\right) = 1 - 2\delta + g(p(N), N)$$
$$\to 1 - 2\delta - \sqrt{\frac{p(N)}{N}}\zeta_i = f(x) \tag{22}$$

for $0 < x \leqslant 1$, $i = [xN]$ and similarly

$$-\xi_i^{(1)}\left(\sum_{j=1, j\neq i}^{N} T_{ij}\xi_j^{(1)} - 2\sum_{j\in\{[(1-\delta)N]+1,\ldots,N\}, j\neq i} T_{ij}\xi_j^{(1)}\right)$$
$$\to -\left(1 - 2\delta - \sqrt{\frac{p(N)}{N}}\zeta_i\right) = -f(x) \tag{23}$$

for $i = [xN]$ and $0 < x \leqslant 1$. So to prove that $\sigma(\xi^{(1)})$ is a fixed point of the dynamics of (1) we use the fact that it is equivalent to checking that

$$f(x) \geqslant 0 \qquad \text{for } 0 \leqslant x \leqslant 1-\delta \qquad \text{and} \qquad -f(x) \geqslant 0 \qquad \text{for } 1-\delta \leqslant x \leqslant 1. \tag{24}$$

Let

$$\alpha(\delta) = \frac{(1-2\delta)^2}{(1-\delta)^2} \tag{25}$$

we see that our claim (24) is true. This proves the 'if' part of our conclusion.

The 'only if': First note the fact that f is a decreasing function of x (see figure 2), $f(1) \to -\infty$ and so to ensure that $\sigma(\xi^{(1)})$ is a fixed point of the dynamics of (1) one must reverse the sign of $\xi_{[xN]}^{(1)}$ starting from a point $x_c(\delta)$. Keeping in mind that one must change the sign of $[\delta N]$ neurons, we finally conclude that $x_c(\delta) = 1-\delta$, $B = \{[(1-\delta)N]+1, \ldots, N\}$ and $\alpha(\delta)$ is given by equation (25).

† More reasonably we should define that $f(1) = -\infty$.

3.3. $\delta_N > 0$ but $\delta_N \to 0$ and $N\sqrt{\delta_N} \to \infty$—retrieval with a decreasing error tolerance

The proof is identical to the case $\delta > 0$ and so we only state results†.

If and only if the configuration defined by

$$\sigma(\xi^{(1)}) := (\xi_1^{(1)}, \xi_2^{(1)}, \ldots, \xi_{[(1-\delta_N)N]}^{(1)}, -\xi_{[(1-\delta_N)N]+1}^{(1)}, \ldots, -\xi_N^{(1)}) \tag{26}$$

is a fixed point of the dynamics of (1), $x_N = 1 - \delta_N$ and

$$1 - 2\delta_N - \sqrt{\alpha_N} d_N = 1 - 2\delta_N - \sqrt{\alpha_N}\Phi^{-1}(1-\delta_N) = 0 \tag{27}$$

which implies that

$$\alpha_N = (1-2\delta_N)^2/(\Phi^{-1}(1-\delta_N))^2. \tag{28}$$

4. Discussion

First of all, let us review our main idea emploited in this paper:

• For stored patterns $\xi^{(\mu)}$, $\mu = 1, \ldots, p$ we derived, by the weak law of large numbers, that the key identity

$$\xi_i^{(1)}\left(\sum_{j=1, j\neq i}^{N} T_{ij}\xi_j^{(1)} - 2\sum_{j\in\{[(1-\delta)N]+1,\ldots,N\}, j\neq i} T_{ij}\xi_j^{(1)}\right) = f(x)$$

for $i = [xN]$, where f is given by equation (21) for $0 < x \leqslant 1$.

• $f(x)$ is a decreasing function of x.

• If we reverse the sign of $\xi_{[xN]}$ with $x > x_c(\delta)$ fulfilling $f(x_c(\delta)) = 0$, then $(\xi_1, \xi_2, \ldots, \xi_{[x_c(\delta)N]}, -\xi_{[x_c(\delta)N]+1}, \ldots, -\xi_N)$ is a fixed point of the dynamics of (1) since the reversement exclusively causes a change of the sign of $f(x)$ for $x > x_c(\delta)$ (see figure).

The crucial step above is to find the exact value of $f(x)$. Fortunately, the extreme value theory in statistics is a ripe file which allows us to carry out our program.

In conclusion, we get a complete picture for the capacity of the Hopfield model for $0 \leqslant \delta < \frac{1}{2}$, i.e.

$$\alpha(\delta) = \begin{cases} 0 & \text{if } \delta = 0 \\ \dfrac{(1-2\delta)^2}{(1-\delta)^2} & \text{if } 0 < \delta < \frac{1}{2} \end{cases} \tag{29}$$

and if $\delta_N \to 0$, $N\sqrt{\delta_N} \to \infty$ (see the following equation (31) and [3 figure 6.5])

$$\alpha_N = \frac{(1-2\delta_N)^2}{(\Phi^{-1}(1-\delta_N))^2}. \tag{30}$$

(i) Formally the capacity α_N coincides with that discovered by Amit *et al* [3] in terms of the replica trick approach since (after omitting higher-order terms)

$$\sqrt{\alpha_N} = 1/(\Phi^{-1}(1-\delta_N))$$

and therefore

$$\delta_N = 1 - \Phi(1/\sqrt{\alpha_N}) = \Phi'(1/\sqrt{\alpha_N})\sqrt{\alpha_N} = \frac{\sqrt{\alpha_N}}{\sqrt{2\pi}} \exp\left(-\frac{1}{2\alpha_N}\right) \tag{31}$$

the very formula appeared in [3, p 969, equation (6.37)] which was obtained in terms of the replica trick calculation.

† The condition $N\sqrt{\delta_N} \to \infty$is only a technical restriction which requires that δ_N goes to zero at a speed that is not too fast. We are certainly more interested in the situation that δ_N tends to zero slowly.

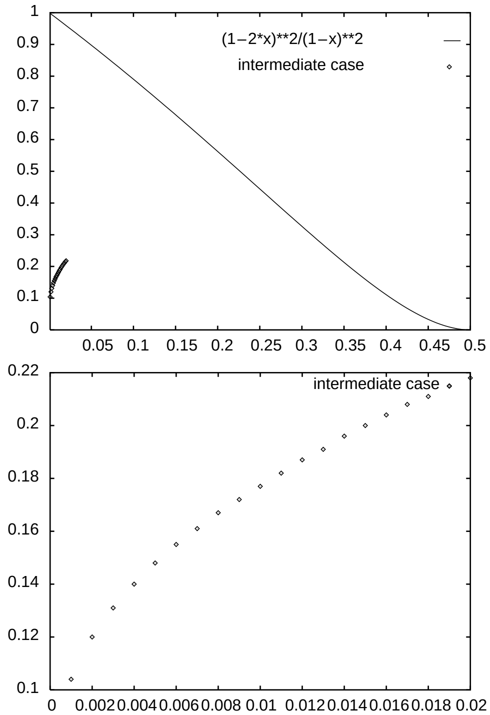

Figure 3. $\alpha(\delta)$ verses δ and α_N verses δ_N (intermediate case, see also figure 6.5 in [3]). $\alpha_N > 0$ is a finite size effect and $\alpha_N \to 0$ when $N \to \infty$.

(ii) The function $\alpha(\delta)$ is a decreasing function of δ. The decreasing accuracy of retrieval memory causes a reduction of the capacity of the Hopfield model. The justification of this fact lies in the fact that we store $p(N)$ patterns in the Hopfield model and these $p(N)$ patterns will *dominate* the network in the sense that they are going to be fixed points of the model. The large errors we introduced the more difficulty we have to store $P(N)$ patterns.

(iii) For given $\delta > 0$ if and only if $p(N) = [\alpha(\delta)N]$ patterns are fixed points of the dynamics of (1). For given $[\alpha N]$ patterns and $\alpha < \alpha(\delta)$ it is always possible for us to add $[(\alpha(\delta) - \alpha)N]$ patterns to the $[\alpha N]$ patterns so that $[\alpha N]$ patterns together with newly added $[(\alpha(\delta) - \alpha)N]$ patterns are all fixed points of the dynamics. In this sense we have

$$\alpha_c(\delta) = \alpha(\delta) \qquad \text{and} \qquad \alpha_c = \sup_{\delta>0} \alpha(\delta) = 1.$$

(iv) It is reasonable to imagine that those neurons taking extreme values are not 'normal' and so we expect a dynamics which does not take into account these states of 'unnormal' neurons to considerably increase the capacity of a network. We are going to discuss this topic in a separate paper.

(v) Here we want to emphasize that our approach (see equations (19) and (26)) presented in this letter also provides a constructive way to find an attractor corresponding to a stored pattern.

(vi) Finally, let us make a comparison between our results and approach and existing results and approach in terms of *the replica trick* [3]. The replica trick approach, a powerful tool and widely used in neural networks, is based upon a few assumptions which are not justified. The *replica trick* in which the value of Z_n (partition function) obtained for integer n is analytically continued for $n \to 0$. Since this limit can also be considered as the derivative of the function $\Phi_N(n) = \frac{1}{N} \log Z_n$ at the point $n = 0$ the analiticity of this function at the point $n = 0$ is requested. The other non-trivial problem is the uniqueness of the limit. The Carleson theorem guarantees that there exists a unique analytic continuation of the function which is known in the positive integer points if this function, $\exp \Phi_N(n)$, has a bound $\exp nC$ as $n \to \infty$. It is easy to see that this condition does not hold for the Sherrington–Kirkpatrick model [9]. In fact recently it was proved that the Parisi solution to the Sherrington–Kirkpatrick spin glass, as applied to more realistic spin glass model, is not valid in any dimension and at any temperature [17]. Our approach, while intuitive and rigorous, conclude that the replica trick approach to the capacity of the Hopfield model is only valid in the case that $\alpha_N \to 0$ (see figure 3). Note that α_N goes to zero as N goes to infinity and $\alpha_N > 0$ is only a phenomenon of finite size effect. Surely a further study to consider would be under what conditions is a fixed point stable, we believe that it would make an interesting topic [12].

Acknowledgment

This paper was partially supported by the A von Humboldt Foundation of Germany, given to JF.

References

[1] Albeverio S, Feng J and Qian M 1995 The role of noises in neural networks *Phys. Rev.* E **52** 6593–606
[2] Amit D, Gutfreund H and Sompolinsky H 1985 Spin-glass models of neural networks *Phys. Rev.* A **32** 1007–18
[3] Amit D 1989 *Modeling Brain Function* (Cambridge: Cambridge University Press)
[4] Balkema A A and De Haan L 1978 Limit distributions for order statistics. I *Theor. Prob. Appl.* **XXIII** 77–92
[5] Balkema A A and De Haan L 1978 Limit distributions for order statistics. II *Theor Prob. Appl.* **XXIII** 341–58
[6] Feng J 1997 Lyapunov functions for neural nets with nondifferentiable input-output characteristics *Neural Computation* **9** 43–9
[7] Feng J and Hadeler K P 1996 Qualitative behaviour of some networks *J. Phys. A: Math. Gen.* **29** 5019–33
[8] Feng J and Tirozzi B 1995 The SLLN for the free-energy of the Hopfield and spin glass model *Helv. Phys. Acta* **68** 365–79
[9] Van Hemmen J L and Palmer R G 1979 *J. Phys. A: Math. Gen.* **12** 563
[10] Herz A V M and Marcus C M 1993 Distributed dynamics in neural networks *Phys. Rev.* E **47** 2155–61
[11] Hopfield J J 1982 Neural networks and physical systems with emergent collective computational abilities *Proc. Natl Acad. Sci., USA* **79** 2554–8
[12] Kinzel W and Opper M 1991 Dynamics of learning *Models of Neural Networks* ed E Domany, J L van hemmen and K Schulten (Heidelberg: Springer) pp 149–71
[13] Komlos J and Paturi R 1988 Convergence results in an autoassociative memory model *Neural Networks* **1** 239–50
[14] Loukianova D 1994 Memory capacity in the Hopfield model *Comptes Rendus De L'Academie Des Sciences Series I-Mathematique* **318** 157–60
[15] Leadbetter M R, Lindgren G and Rootzén H 1983 *Extremes and Related Properties of Random Sequences and Processes* (Berlin: Springer)
[16] Newman C 1988 Memory capacity and neural network models: Rigorous lower bounds *Neural Networks* **1** 223–38
[17] Newman C and Stein D L 1996 Non-mean-field behaviour of realistic spin glass *Phys. Rev. Lett.* **76** 515–8
[18] MacEliece R J, Posner E C, Rodemich E R and Venkatesh S S 1987 The capacity of the Hopfield associative memory *IEEE Trans. Inform. Theor.* **33** 461–82
[19] Pastur L, Shcherbina M and Tirozzi B 1994 The replica symmetric solution of the Hopfield model without replica trick *J. Stat. Phys.* **74** 1161–83
[20] Pastur L, Shcherbina M and Tirozzi B 1997 Replica symmetric solutions *Proc. WIAS Workshop on Mathematics of Spin Systems with Random Interactions* ed A Bovier (Berlin: Springer) in press
[21] Watts V, Rootzén H and Leadbetter M R 1982 Convergence in distribution of intermediate order statistics from stationary sequences *Ann. Prob.* **10** 653–62